On step-by-step approximation of linear delay differential equation

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ABSTRACT

A new numerical method of solving linear delay differential equations based on the concept of step-by-step approximation of the system equation is formulated. The solution is presented in the of n^{th} finite series of k^{th} order for n + 1-subinterval. Its application to numerical problems confirmed its suitability.

INTRODUCTION

It is common that most dynamics encountered in physical life express the states of situation. A more realistic system should encompass not only the present states (t) but also the past states (t-r) of the system. This principle appears to permeate various aspects of life and has of late influenced many researches.

A differential equation, which involved the present states as well as the past states of any physical system, is known as delay differential equation (functional differential equation); Delay differential system are further divided into two broad types:(a) retarded functional differential equations, where delay is only in the states of the system.

$$\frac{dx(t)}{dt}=x(t-r), r>0 \quad (1)$$

$$x(t_0) = x_0 = \varphi_o;\ s\varepsilon(t-r)$$

and (b) neutral functional differential equations; delays are in the states as well as in the derivative,

$$\frac{d(x(t) - x(t-r))}{dt}=x(t-r), r>0 \quad (2)$$

$$x(t_0) = x_0 = \varphi_o;\ s\varepsilon(t-r)$$

A unique solution of systems (1) and (2) are not easily come by, unlike the ordinary differential equation.

But, the establishment of conditions for existence and uniqueness of solution of equations (1) and (2) by Hale and Verduyn (1993), Driver (1995) and Lambert(2004) confirmed that the system solution exists and are unique. A founder-like analysis of the existence of the solution approached by Wright (1999) and Falbo (1998) also confirmed the existence and uniqueness theory on solutions of functional delay differential system.

Several concepts have been formulated to find an analytic solution of the functional delay differential system by solving the characteristics equation. The major setback has been on their special transcendental characters, which result in solutions being expressed in an infinite series form. Lam (1994) considered the connections between the theory and problems of the system. In his view, not the entire series solution but few can characterize the system problem. Therefore, he employed the Pade approximation which results in a shortened repeating fraction for the approximation of the characteristic equation. Ash and Ulsoy (2003) and Ulsoy and Ash(2005) solved the transcendental characteristic equations (1) and (2) by using a class of functions known as Lambert functions (Lambert 2004), with the limitation that a delay functional equation that does not satisfy the Lambert functions cannot be solved.

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Solution of linear delay differential equation

This paper presents a new numerical approach which is an approximating concept based on step-by-step solving of the system on each interval. The advantage of this approach is that the approximate solution obtained is analogous to the general solution of the ordinary differential equations.

PROBLEM STATEMENT

Consider a general delay system of the form:

$$\dot{x}(t) = ax(t-r), \quad x(t_\theta) = \varphi_0 \quad r > 0 \tag{3}$$

Consider equation (3) on the x_n sub-interval:

x_1; $t_o - r \le t \le t_o$

x_2; $t_o \le t \le t_o + r$

x_3; $t_o + r \le t \le t_o + 2r$

x_4; $t_o + 2r \le t \le t_o + 3r$

x_n; $t_o + (n-1)r \le t \le t_o + nr$

such that $r > 0$ and $t \; \varepsilon \; (t_o - r, t_o + nr)$.

Mathematical formulation

Step by step analysis of equation (3) follows:

$$\dot{x}(t) = ax(t-r); \qquad t - r_o \le t \le t_o \tag{4}$$

Defining the interval of t in terms of the delay variable (r).

$$t_o - 2r \le t - r \le t_o - r$$

then

$$x_1(t) = \varphi_o - ax(t-r)$$

$$x_1(t) = \varphi_o - ax(t-r)t + c$$

Solving for c using equation(3), we get that

$$c = \varphi_o - \varphi_o(t_o - r) + ax(t_o - r)(t_o - r)$$

Therefore, $x_1(t) = \varphi_o$ (5)

For $x_2(t)$; $t_o \le t_o \le + r$

this implies that $t_o - r \le t - r \le t_o$

such that

$$x(t-r) = t_o$$

$$\dot{x}_2(t) = \varphi_o - a\varphi_o$$

$$x_2(t) = \varphi_2(t) - a\varphi_o(t) + c$$

From equation(5), we can see that

$$c = \varphi_o - \varphi_o(t_o) + a\varphi_o(t_o)$$

Therefore,

$$X_2(t) = \varphi_o - a\varphi(t - t_o) \tag{6}$$

For $x_3(t)$; $t_o + r \le t \le t_o + 2r$, this implies that

$$t_o \le t - r \le to + r$$

such that $x(t-r) = t_o + r$

$$x_3(t) = \varphi_o + a(\varphi_o \, a\varphi\,(t - t_o))$$

$$x_3(t) = \varphi_o(t) + a\varphi_o(t) - a^2\varphi\left(\frac{t^2}{2} - t_o t\right) + c;$$

From equation(6) we can see that

$$c = \varphi_o - a\varphi(t - t_o) - \varphi_o(t_o - r) - a\varphi(t_o - r) + a^2\varphi\left(\frac{(t_o+r)^2}{2} - t_o(t_o - r)\right)$$

Therefore,

$$x_3(t) = \varphi_o + a\varphi_o(t - t_o) + a^2\varphi\left(\frac{t^2 - (t_o+r)^2 - 2(t_o+r)^2 - 2t_o t}{2!}\right) \tag{7}$$

for $x_4(t)$; $t_o + 2r \le t \le t_o + 3r$ this implies that $t_o + r \le t\text{-}r \le t_o + 2r$ such that $x(t-r) = t_o + 2r$

$$x_4(t) = \varphi_o + a\left[\varphi_o + a\varphi(t - t_o) + a^2\varphi\left(\frac{t^2}{2} - \frac{(t_o+r)^2}{2} - (t_o + r)^2 - tt_o\right)\right]$$

$$x_4(t) = \varphi_o(t) + a\varphi_o(t) - a^2\varphi\left(\frac{t^2}{2} - t_o t\right) + a^3\varphi\left(\frac{t^3}{3!} - \frac{(t_o+r)^2 t}{2} - \frac{t^2 t_o}{2}\right) + c$$

From equation(7) we can see that

$$c = \varphi_o - a\varphi(t - t_o) + a^2\varphi\left(t^2 - \frac{(t_o+r)^2}{2} - (t_o+r)^2 - tt_o\right)$$

$$- \varphi_o(t) - a\varphi_o(t) + a^2\varphi\,\frac{(t^2 - t_o t)}{2} - a^3\varphi\left(\frac{t^2}{3!} - \frac{(t_o+r)^2 t}{2} - \frac{t^2 t_o}{2}\right)$$

Therefore,

$$x_4(t) = \varphi_o + a\varphi(t - t_o) + a^2\varphi\left(\frac{(t^2 - 3(t_o+r) + 2(t_o - tt_o)}{2!}\right) +$$

$$^2\varphi\,\frac{(t^3 - t_o + 2r)^3}{3!} - (t_o + r)^2 t - t^2 t_o) \quad (8)$$

NUMERICAL APPLICATION

$\dot{x}(t) = +x(t-1)$

$x(t) = 1;\ 0 \leq t \leq 1$ (9)

Consider equation (9) on the interval $0 \leq t \leq 1$;

then

$\dot{x}(t) = 1$ (10)

On $1 \leq t \leq 2$ defining t in terms of the delay $0 \leq t - 1 \leq 1$

such that $x(t - 1) = 1$

$\dot{x}(t) = x_1(t) + x(t-1)$

$x(t) = 2t + c$

Solving for c, using equation (10) at t = 1,

we get that

$x_2(t) = 2(t - 1) + 1$ (11)

$x_2(t - 1) = 2(t - 2) + 1$

On $2 \leq t \leq 3$, this implies that $1 \leq t - 1 \leq 2$ (12)

such that $x(t - 1) = 2$

$\dot{x}(t) = x_1(t - 1) + x_2(t - 1)$

$\dot{X}(t) = 2 + 2(t - 2)$

$x(t) = 2t + (t - 2)^2 + c$

Solving for c using equation (11) at t – 2

$x_3(t) = 1 + 2(t - 1) + (t - 2)^2$ (13)

$x_3(t - 1) = 1 + 2(t - 2) + (t - 3)^2$ (14)

On $3 \leq t \leq 4$ this implies that $2 \leq t - 1 \leq 3$

such that $x(t - 1) = 3$

$\dot{X}(t) = x_1(t) + x_3(t - 1)$

$\dot{X}(t) = 2t + 2(t - 2) + (t - 3)^2$

$$x(t) = 2t + (t-2)^2 + \frac{(t-3)^3}{3} + c$$

Solving for c, using equation (13) at t = 3

$$x_4(t) = 1 + 2(t-1) + (t-2)^2 + \frac{(t-3)^3}{3}$$

$$\therefore x_4(t-1) = 2(t-2) + (t-3)^2 + \frac{(t-4)^3}{3} \quad (15)$$

On $n \leq t \leq n + 1$

$$x(t) = 1 + 2(t-1) + \frac{2(t-2)^3}{2!} + \frac{(t-3)^3}{3!} + \ldots + \frac{2(t-n)^n}{n!}$$

The general series form of x(t) is

$$X(t) = 1 + 2\sum_{k=0}^{m+1}\frac{(t-k)^k}{k!}; \quad k > 0 \quad (16)$$

CONCLUSION

An approximate solution of a functional linear delay differential equation is obtained for each n + 1 – subinterval. The advantage of this method is that behavior of solutions such as stability analysis on each subinterval can be formulated. Also, solutions obtained are comparable to the general solution of the ordinary differential equation.

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